

# Waves and energy in chiral nihility

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## Abstract

A model for a chiral material in which both the permittivity and permeability are equal to zero is discussed. Such a material is referred by us as a “chiral nihility”. It is shown that this exotic material can be realized as a mixture of small helical inclusions. Wave solutions and energy in such a medium are analyzed. It is shown that an extraordinary wave in chiral nihility is a backward wave. Wave reflection and refraction on a chiral nihility interface is also considered. It is shown that a linearly polarized wave normally incident onto this interface produces the wave of “standing phase” and the same wave in the case of oblique incidence causes two refracted waves, one of them with an anomalous refraction.

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# 1 Introduction

Recently, a lot of attention has been paid to composite materials which in certain frequency regions can be described by effective permittivity and permeability having negative real parts (or, approximately, by real negative material parameters). In this respect, A. Lakhtakia conceptually considered two-phase mixtures of an ordinary isotropic material with positive material parameters and of a double negative material, such that the effective parameters of the mixture become null. He introduced term “nihility” for such medium, whose  $\epsilon = 0$ ,  $\mu = 0$  [1]. He also used this notion in a later paper [2] when discussing the concept of perfect lens proposed by J.B. Pendry [3]. Lakhtakia’s conclusion was that nihility material is not physically realizable.

In this paper, we model a regular array of small chiral (or  $\Omega$ -shaped) ideally conducting particles and find out that at a certain frequency this system behaves as an effective media with null-valued permittivity and permeability. Of course, in real physical situations there will be nonzero imaginary parts of the material parameters due to absorption in the particles. However, no waves can travel in nonchiral nihility media [1], since the field equations reduce to the static ones.

Furthermore, in this paper, we will generalize the concept of nihility and introduce a more general concept of *chiral nihility* composite materials. These materials can be realized in a similar way as mixture with “ordinary” nihility. We have to add chiral inclusions, for example, all of the same handedness. At the frequency where the real parts of both permittivity and permeability become zeros, the chirality parameter is nonzero, and, as we will demonstrate by a numerical example, the imaginary parts of all the parameters can be rather small compared to the chirality parameter. In these media, waves can propagate, and the material exhibits some very interesting properties. In particular, double refraction takes place at an interface between free space and an *isotropic* chiral nihility. A linearly polarized wave is split into two circularly polarized once, and one if these two components suffers negative refraction, as in backward-wave or double negative materials.

The constitutive relations for isotropic chiral media read [4]

$$\mathbf{D} = \epsilon\epsilon_0\mathbf{E} - j\kappa\sqrt{\epsilon_0\mu_0}\mathbf{H} \quad (1)$$

$$\mathbf{B} = j\kappa\sqrt{\epsilon_0\mu_0}\mathbf{E} + \mu\mu_0\mathbf{H} \quad (2)$$

Chiral nihility media are, by definition, media with the material parameters satisfy, at a certain frequency  $\omega_0$ ,

$$\epsilon = 0, \quad \mu = 0, \quad \kappa \neq 0 \quad (3)$$

Thus, the material relations reduce to

$$\mathbf{D} = -j\kappa\sqrt{\epsilon_0\mu_0}\mathbf{H} \quad (4)$$

$$\mathbf{B} = j\kappa\sqrt{\epsilon_0\mu_0}\mathbf{E} \quad (5)$$

The different electric and magnetic units call for a renormalization of the quantities when there is magnetoelectric coupling. Also, to achieve a compact notation for the material response analysis, a good technique is the six-vector notation with which the constitutive

parameters are contained in a material matrix  $\mathbf{M}$ :

$$\begin{pmatrix} c\eta_0\mathbf{D} \\ c\mathbf{B} \end{pmatrix} = \begin{pmatrix} \epsilon & -j\kappa \\ j\kappa & \mu \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \eta_0\mathbf{H} \end{pmatrix} = \mathbf{M} \begin{pmatrix} \mathbf{E} \\ \eta_0\mathbf{H} \end{pmatrix} \quad (6)$$

in other words,

$$\mathbf{d} = \mathbf{M}\mathbf{e} \quad (7)$$

where the fields and displacements now carry the same dimensions (V/m), and the material matrix components are dimensionless.<sup>1</sup>

## 2 Model of the material parameters of chiral nihility

Using the antenna model of canonical chiral particles that consist of a small loop connected to a short wire dipole antennas [5] and the Maxwell Garnett mixing rule, we can calculate the effective parameters of an isotropic array of chiral particles. In these calculations, we assume that the array is regular, so that the scattering loss of the particles is suppressed by the interaction field. The absorption loss has been taken into account by adding appropriate real parts to the input impedances of the loop and wire portions of the canonical chiral helix. The geometrical parameters have been chosen so that the real parts of the permittivity and permeability become zero at the same frequency. The results of these calculations are shown in Figures 1 and 2. The second picture is a blow-up near the frequency of zero permittivity and permeability. The inclusion sizes are the following: the arm length of the straight dipole  $l = 2.7$  mm, the loop radius  $a = 2.45$  mm, the wire radius  $r_0 = 0.25$  mm. Wires are made of copper with conductivity  $\sigma = 5.8 \cdot 10^7$  S/m, and the volume fraction is  $f = 0.2$ . The last value is defined by introducing imaginary spheres of the minimum radius totally enclosing the helices.

As is obvious from the graphs in Figures 1 and 2, at the frequency equal to 6.38 GHz, both the permittivity and permeability of the material have zero real parts. The chirality parameter is about 0.7 at the frequency. The imaginary parts are rather small compared with the chirality parameter near this frequency and we will neglect them in the following analysis. In passing, let us note that a racemic arrangement of the same particles gives a *nihility* material with  $\epsilon = 0$ ,  $\mu = 0$ , and  $\kappa = 0$ . Therefore, not only nihility medium is possible but also a material displaying chiral nihility.

Let us next move on to study theoretical limitations for the material parameters in complicated media.

## 3 Material parameter restrictions

The limitation for material parameters in lossless chiral media are [4]

$$\kappa^2 \leq \epsilon\mu \quad (8)$$

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<sup>1</sup>The coefficients in the renormalization are the vacuum constants  $c = 1/\sqrt{\epsilon_0\mu_0}$  and  $\eta_0 = \sqrt{\mu_0/\epsilon_0}$ .

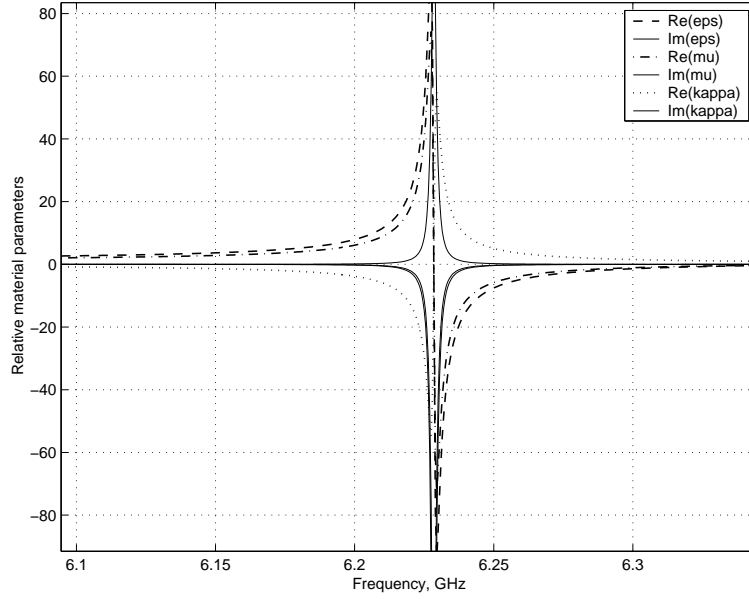


Figure 1: Effective material parameters of a lattice of canonical helices. See text for the details of the mixture.

This restriction comes from the requirement that the wave numbers  $k_{\pm} = k_0(\sqrt{\mu\epsilon} \pm \kappa)$  (for lossless media) should be positive. However, why is it necessary to have both wave vectors positive? The answer to this is not obvious. Another way to justify the restriction (8) is to consider the eigenvalues of the material matrix. These are—being the solutions for the equation

$$\mathbf{M}\mathbf{e} = \lambda\mathbf{e} \quad (9)$$

for the eigenproblem—the following two values

$$\lambda_{1,2} = \frac{\epsilon + \mu}{2} \pm \sqrt{\left(\frac{\epsilon - \mu}{2}\right)^2 + \kappa^2} \quad (10)$$

Now, the usually accepted limitations (8) are seen to correspond to the requirement that

$$\lambda_1 \geq 0, \quad \lambda_2 \geq 0 \quad (11)$$

In other words, the limitation means that the material matrix  $\mathbf{M}$  be positive definite. This, subsequently, corresponds to the condition that the sum of electric and magnetic energy densities is positive:

$$W_e + W_m \propto \mathbf{E} \cdot \mathbf{D} + \mathbf{H} \cdot \mathbf{B} \propto \mathbf{e} \cdot \mathbf{M}\mathbf{e} \quad (12)$$

With this in mind it seems that the limitation (8) is rather reasonable. But locally we can certainly violate the limitation. The Veselago medium, or a basic plasma medium for example, as such violate against it. Should we have  $\epsilon \leq 0$ , the property is sufficient to cause locally negative static energy density which is proportional to  $\epsilon|\mathbf{E}|^2$ . Because plasma is a well-documented real-world phenomenon, and also a 2D analogue of Veselago

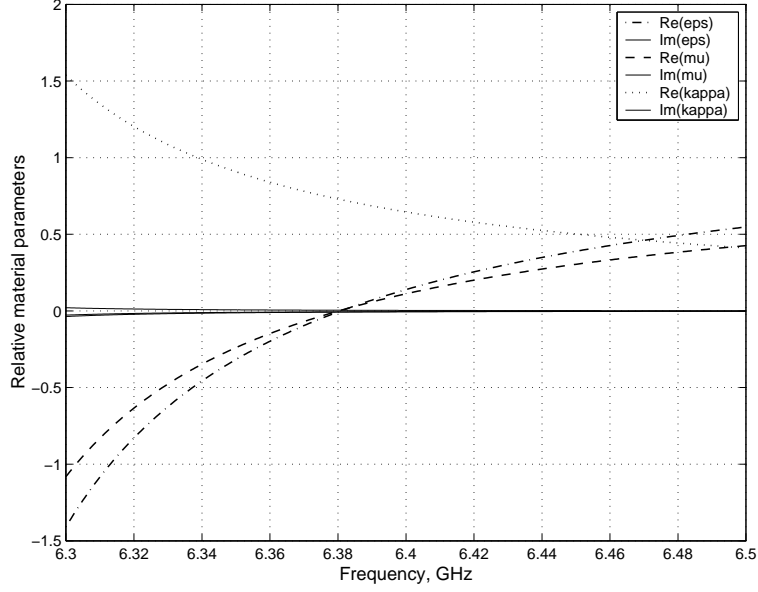


Figure 2: The same as in Figure 1 for a narrow frequency range.

media has been fabricated in laboratories [8], we have to conclude that the restriction is too tight. In fact, the nihility medium as well as Veselago medium possess strong frequency dispersion, and the energy density should be calculated with this in mind [6]. The time-averaged energy density in a dispersive chiral medium with negligible losses is expressed as [7]

$$\langle W \rangle_t = \frac{1}{4} \left[ \mathbf{E}^* \cdot \frac{\partial(\omega\epsilon)}{\partial\omega} \cdot \mathbf{E} + \mathbf{H}^* \cdot \frac{\partial(\omega\mu)}{\partial\omega} \cdot \mathbf{H} + \frac{2}{c} \text{Im} \left( \mathbf{E}^* \cdot \frac{\partial(\omega\kappa)}{\partial\omega} \cdot \mathbf{H} \right) \right] \quad (13)$$

We will come back to the discussion of the energy density after considering possible wave solutions in chiral nihility.

On the other hand, the requirement for the imaginary parts of the material parameters for any medium

$$\text{Im} \{ \epsilon \} \leq 0, \quad \text{Im} \{ \mu \} \leq 0, \quad \text{Im} \{ \kappa \} \leq \text{Im} \{ \sqrt{\epsilon\mu} \} \quad (14)$$

should hold, because the amplitude of a propagating wave should not grow exponentially in a dissipative medium. And this is indeed the case. According to the model discussed and applied in the previous section, we can see that the imaginary parts of the material constants satisfy (14). Therefore also  $\text{Im}\{k\} < 0$  which is necessary in order that the wave propagation according to  $\exp(-jkz)$  is not exponentially increasing.

## 4 Waves in chiral nihility materials

Let us next analyze waves in chiral nihility media. Taking the time dependence as  $\exp(j\omega t)$ , the Maxwell equations for chiral nihility can be written as

$$\begin{aligned} \nabla \times \mathbf{E} &= k_0 \kappa \mathbf{E} \\ \nabla \times \mathbf{H} &= k_0 \kappa \mathbf{H}, \end{aligned} \quad (15)$$

where  $k_0 = \omega/c$  is the wavevector in vacuum. Solutions of (15) are the eigenvectors of the curl operator and describe the circularly-polarized waves with helicity parameter  $k_0\kappa$ . At first look it seems that each of equations (15) determines independent “electric” and “magnetic” waves, like Langmuir waves in plasmas. However, we have to remember that the chiral nihility medium is obtained by the mixture of components having positive and negative electric and magnetic polarizabilities which must be dispersive. Thus, we have to adopt that electric and magnetic field are not independent but are connected via the wave impedance

$$\eta = \eta_0 \lim_{\epsilon \rightarrow 0, \mu \rightarrow 0} \sqrt{\frac{\mu}{\epsilon}} \quad (16)$$

Of course in nihility the quotient of impedance has to be carefully defined as a limit process. The limit  $\mu/\epsilon$  for  $\omega \rightarrow \omega_0$  depends on the behavior  $\epsilon(\omega), \mu(\omega)$  in the vicinity of  $\omega_0$ . In our particular example (Figure 2) this value is close to the impedance of free space, because the values of the effective permittivity and permeability are close in the vicinity of the nihility point.

Assuming the wave propagation direction is along  $z$ -axis, solutions of equations (15) can be written in form

$$\begin{aligned} E_x &= e_0 \exp(\mp j k_0 \kappa z) & H_x &= \pm j e_0 \eta^{-1} \exp(\mp j k_0 \kappa z) \\ E_y &= \mp j e_0 \exp(\mp j k_0 \kappa z) & H_y &= e_0 \eta^{-1} \exp(\mp j k_0 \kappa z) \\ E_z &= 0 & H_z &= 0 \end{aligned} \quad (17)$$

where  $\eta$  is the wave impedance of chiral nihility (16) and the two signs correspond to the phase advance direction along the positive or negative direction of axis  $z$ . Obviously, the waves are circularly polarized with the opposite sense of rotation for the opposite propagation directions. The waves in chiral nihility propagate with the propagation constant  $\beta = \pm k_0 \kappa$  and the phase velocity  $v_p = \pm c/\kappa$ , where  $c$  is the speed of light. The phase velocity  $v_p$  can be either subluminal if  $\kappa > 1$  or superluminal, if  $\kappa < 1$ .

Let us next discuss the group velocity of plane waves in chiral nihility. In usual chiral media there exist two eigenwaves traveling along the positive  $z$ -direction with the phase constants  $\beta_{1,2} = k_0(\sqrt{\epsilon\mu} \pm \kappa)$ . For the inverse group velocity of these waves we obtain

$$\frac{1}{v_{gr1,2}} = \frac{\partial \beta_{1,2}}{\partial \omega} = \frac{\sqrt{\epsilon\mu}}{c} + \frac{\omega}{c} \frac{\partial(\sqrt{\epsilon\mu})}{\partial \omega} \pm \left( \frac{\kappa}{c} + \frac{\omega}{c} \frac{\partial \kappa}{\partial \omega} \right) \quad (18)$$

Usually, both these values are positive, and both eigenwaves are usual forward waves. The other two eigensolutions have the oppositely directed phase vectors ( $\beta_{1,2} = -k_0(\sqrt{\epsilon\mu} \pm \kappa)$ ) and, of course, their group velocities are negative of that given by (18). At the cross-over point where  $\sqrt{\epsilon\mu} = 0$  the eigensolutions become degenerate in terms of the phase constants: we have only *two* waves with  $\beta = \pm k_0 \kappa$  instead of four. However, the group velocities are still given by (18) with  $\sqrt{\epsilon\mu} = 0$ :

$$\frac{1}{v_{gr1,2}} = \frac{\omega}{c} \frac{\partial(\sqrt{\epsilon\mu})}{\partial \omega} \pm \left( \frac{\kappa}{c} + \frac{\omega}{c} \frac{\partial \kappa}{\partial \omega} \right) \quad (19)$$

and there are still *four* different values for the group velocity of four eigenwaves (the other two solutions are the negative of (19)). If we assume that at this special point both values

(19) are positive, as is generally the case in chiral media, then the conclusion is that one of the two eigenwaves with positive group velocity has a negative phase velocity, that is, one of the eigenwaves is a backward wave.

Let us consider the energy characteristics of the waves in chiral nihility. The time-averaged Poynting vector can be written as [6]:

$$\langle \mathbf{P} \rangle_t = \frac{1}{4}(\mathbf{E} \times \mathbf{H}^* + \mathbf{E}^* \times \mathbf{H}) \quad (20)$$

Substituting the fields (17) to (20), we obtain

$$\langle \mathbf{P} \rangle_t = e_0^2 Y \quad (21)$$

where  $Y = \eta^{-1}$  is the wave admittance, which is real for chiral nihility.

Assuming  $\epsilon = 0$ ,  $\mu = 0$  in the expression for the averaged energy density (13) and substituting the fields (17) into (13), one obtains

$$\langle W \rangle_t = \frac{e_0^2}{2} \left[ \omega \frac{\partial \epsilon}{\partial \omega} + Y^2 \omega \frac{\partial \mu}{\partial \omega} + \frac{2Y}{c} \left( \kappa + \omega \frac{\partial \kappa}{\partial \omega} \right) \right] \quad (22)$$

Since the following holds

$$\frac{\partial(\sqrt{\epsilon\mu})}{\partial \omega} = \frac{\eta}{2} \left( \frac{\partial \epsilon}{\partial \omega} \pm Y^2 \frac{\partial \mu}{\partial \omega} \right) \quad (23)$$

we can conclude that the energy velocity

$$v_{en} = \frac{\langle P \rangle_t}{\langle W \rangle_t} \quad (24)$$

coincides with the group velocity, expressed by equation (19).

In contrast to the usual chiral medium, where limitations for material parameters (8) take place and both of right-hand and left-hand circularly polarized waves are forward ones, in chiral nihility the left-hand polarized wave is the backward wave (here we assumed  $\kappa > 0$ ).

## 5 Wave reflection and refraction on a chiral nihility interface

Let us consider a linearly polarized plane wave illuminating an interface between vacuum  $\epsilon_1 = \mu_1 = 1$  and chiral nihility  $\epsilon_2 = \mu_2 = 0$ ,  $\kappa \neq 0$ . In the case of the normal incidence such an incident wave excites right-hand and left-hand polarized waves in chiral nihility, having equal amplitudes, the same directions of the energy flow (away from the source), and the opposite phase velocities. If the wave impedance in chiral nihility is close to that of free space, the reflected wave can be neglected. The total transmitted electric field can be written as

$$\begin{aligned} \mathbf{E} &= \mathbf{E}_+ + \mathbf{E}_- \\ &= \frac{e_0}{2}(\mathbf{x}_0 - j\mathbf{y}_0) \exp(-j\kappa k_0 z) + \frac{e_0}{2}(\mathbf{x}_0 + j\mathbf{y}_0) \exp(j\kappa k_0 z) \\ &= e_0(\mathbf{x}_0 \cos \kappa k_0 z - \mathbf{y}_0 \sin \kappa k_0 z) \end{aligned} \quad (25)$$

Thus, the total electric field represents the wave of “standing phase”, whose amplitude changes along  $z$ -axis in accordance with (25). Such a wave has infinite phase velocity, but its energy velocity is expressed by equation (24). It can be easily shown that the proper contributions to the Poynting vector from refracted left-hand and right-hand waves have the same signs and their mutual (interference) time-averaged left-hand-right-hand contribution is equal to zero. The field forms a standing spiral structure in space.

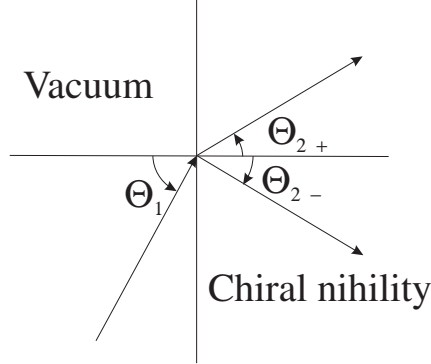


Figure 3: An incident plane wave hits the boundary of an isotropic chiral nihility half space. Note the two refracted rays. The refracted waves are circularly polarized, with the opposite sense of rotation. The arrows show the directions of the power flow.

Next, let us consider oblique incidence of a linearly polarized plane wave from vacuum half space (region 1) on an interface with a chiral nihility half space (region 2) according to Figure 3. In the medium, there will propagate two circularly-polarized waves. The Snellius law reads [4]

$$\sin \theta_{2\pm} = \frac{1}{\pm\kappa} \sin \theta_1 \quad (26)$$

This means that the two refracted waves propagate at the angles  $\theta_{2\pm}$ ,  $\theta_{2-} = -\theta_{2+}$  to the normal. Anomalous refraction takes place for the left-hand polarized wave, because that wave is a backward-wave. We can conclude that in this case double refraction is possible at an interface between two isotropic media. Moreover, one of the rays suffers anomalous refraction, like in the Veselago medium. One can say that one of the eigenwaves in chiral nihility sees an equivalent isotropic dielectric, but the other one sees a backward-wave (Veselago) medium.

## 6 Concluding remarks

The present paper has introduced the concept of chiral nihility medium. Using the model for a regular lattice of complex particles, a composite was modeled that displays parameters required for chiral nihility. The parameters of the material were found to be consistent with the physical restrictions for complex media. Analyzing the field equations, we have introduced the concept of wave impedance in nihility materials defined through a limit of the permeability/permittivity ratio. Also, wave propagation and refraction involving chiral



nihility media have been discussed. It has been found that the eigenwaves are circularly polarized, like in isotropic chiral media, but one of the eigenwaves is a backward-wave, like in Veselago media with double negative parameters. Note that materials with negative parameters are sometimes called “left-handed” materials (because the triplet of vectors  $\mathbf{E}$ ,  $\mathbf{H}$ , and  $\mathbf{k}$  is left-handed). This name can be especially confusing in the present case of chiral nihility, where the left- or right-hand circularly polarized wave is a backward wave. As such, handedness of chiral materials with helical inclusions has nothing in common with the existence of backward waves in Veselago media. We have found that an interface between an isotropic chiral nihility material and free space has a very interesting property of double refraction: the wave is split into two circularly polarized components, such that one of them is refracted positively, but the other one is refracted negatively, like in Veselago media. A helical standing wave pattern is formed in chiral nihility material, if it is excited by a normally incident plane wave.

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